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Umbral Calculus, Bailey Chains, and Pentagonal Number Theorems

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DEDICATED TO THE MEMORY OF GIAN-CARLO ROTA

We revisit the umbral methods used by L. J. Rogers in his second proof of the Rogers–Ramanujan identities. We shall study how subsequent methods such as the Bailey chains and their variants arise naturally from Rogers' insights. We conclude with the introduction of multi-dimensional Bailey chains and apply them to prove some new Pentagonal Number Theorems. © 2000 Academic Press

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1. INTRODUCTION

In the summer of 1998, I attended the conference, Combinatorics and Physics '98, held at Los Alamos and organized by Bill Chen and Jim Louck. One day, Gian-Carlo invited me and several others to lunch at a Mexican restaurant in a small town many miles from Los Alamos. During the drive there, he and I discussed at length Problem 4, A Unified Theory of Special Functions, in his paper Ten Mathematics Problems I Will Never Solve [11]. As we discussed this, I mentioned my belief that L. J. Rogers' second proof of the Rogers–Ramanujan identities [10; Sect. 1] can be viewed as an umbral calculus method that has its origins in the notorious "eighteen papers of Liouville" [5, Chap. XI] which were explained umbrally by Humbert [7]. Furthermore, I speculated, an analysis of subsequent work viewed through the umbral calculus lens should provide an overarching account of Bailey Chains [2] and their extensions (e.g., [4, 9]). He seemed intrigued and told me to be sure to write it up and send

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it to him. I said that, of course I would. I put the project on a back burner, and it remained there until the spring of 1999 when the earthshaking news arrived: Rota had died. Tragic and unbelievable news!

The invitation to prepare a paper for this volume brought our Los Alamos conversation back to mind vividly. In Sections 2 and 3, I shall provide the details of the project I outlined in my conversation with Gian-Carlo.

The genesis of all this, of course, is (as I said above) Rogers' second proof of the Rogers–Ramanujan identities. Consequently, the work in Sections 2 and 3 will be justified only if we are led to something really new. Section 4 will describe some new results. To maintain reasonable brevity we shall restrict the discussion to some new Pentagonal Number Theorems.

Recall the famous Pentagonal Number Theorem of Euler [1; p. 11]:

$$1 = \frac{\sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2}}{\prod_{n=1}^{\infty} (1-q^n)}. \tag{1.1}$$

In Section 4 we shall prove

$$\sum_{n=1}^{\infty} \frac{q^{2n^2}}{(q;q)_{2n}} = \frac{\sum_{n,m=-\infty}^{\infty} (-1)^{n+m} q^{n(3n-1)/2+m(3m-1)/2+nm}}{\prod_{n=1}^{\infty} (1-q^n)^2}, \tag{1.2}$$

and

$$\sum_{i,j,k\geqq 0} \frac{q^{i^2+j^2+k^2}}{(q;q)_{i+j-k}(q;q)_{i+k-j}(q;q)_{j+k-i}} = \frac{\sum_{n,m,p=-\infty}^{\infty} (-1)^{n+m+p} q^{n(3n-1)/2+m(3m-1)/2+p(3p-1)/2+nm+np+mp}}{\prod_{n=1}^{\infty} (1-q^n)^3}. \tag{1.3}$$

We conclude with a discussion of possible extensions of the ideas introduced here.

2. ROGERS' SECOND PROOF

We now provide a variation of L. J. Rogers' second proof of the Rogers–Ramanujan identities [10; Sec. 1]. The starting point is Jacobi's triple product identity [6; p. 239, Eq. (II.28)]

$$\sum_{n=-\infty}^{\infty} q^{n^2} z^n = (q^2;q^2)_\infty (-zq;q^2)_\infty (-z^{-1}q;q^2)_\infty, \tag{2.1}$$

where

$$(A;q)_N = \prod_{n=0}^{\infty} \frac{(1-Aq^n)}{(1-Aq^{n+N})}. \tag{2.2}$$

To (2.1) we shall apply Euler's identity [6; p. 236, Eq. (II.1)]

$$\sum_{n=0}^{\infty} \frac{q^{n^2} z^n}{(q^2;q^2)_n} = (-zq;q^2)_\infty. \tag{2.3}$$

Hence

$$\frac{\sum_{n=-\infty}^{\infty} q^{n^2} z^n}{(q^2;q^2)_\infty} = \sum_{r,s \geqq 0} \frac{z^{r-s} q^{r^2+s^2}}{(q^2;q^2)_r (q^2;q^2)_s}. \tag{2.4}$$

We now invoke an umbral mapping:

$$z^n \to \begin{cases} 0 & \text{if } n \text{ odd} \\ \alpha_{\frac{n}{2}} & \text{if } n \text{ even.} \end{cases} \tag{2.5}$$

Consequently

$$\frac{\sum_{n=-\infty}^{\infty} q^{4n^2} \alpha_n}{(q^2;q^2)_\infty} = \sum_{\substack{r,s \geqq 0 \\ r \equiv s \pmod 2}} \frac{\alpha_{\frac{r-s}{2}}\, q^{r^2+s^2}}{(q^2;q^2)_r (q^2;q^2)_s}. \tag{2.6}$$

The double sum on the right may be simplified if we rewrite the indices using $r = m + j$ and $s = m - j$ (which is legitimate because r and s have the same parity). If in addition we replace α_s by α_s / q^{2s^2} and then replace q by $q^{1/2}$, we find that

$$\frac{1}{(q;q)_\infty} \sum_{n=-\infty}^{\infty} q^{n^2} \alpha_n = \sum_{m \geqq 0} q^{m^2} \beta_m, \tag{2.7}$$

where

$$\beta_m = \sum_{j=-m}^{m} \frac{\alpha_j}{(q;q)_{m-j} (q;q)_{m+j}}. \tag{2.8}$$

The first Rogers–Ramanujan identity follows once one proves that if

$$\alpha_j = (-1)^j q^{j(3j-1)/2}, \tag{2.9}$$

then

$$\beta_m = \frac{1}{(q;q)_m}. \tag{2.10}$$

The fact that (2.8) is fulfilled for these particular α_j and β_m is proved by mathematical induction (cf. [3; pp. 22–23]), and we shall not repeat the proof here. Once this has been established one is allowed to conclude that (2.7) also holds; therefore

$$\frac{1}{(q;q)_\infty} \sum_{n=-\infty}^{\infty} (-1)^n q^{n(5n-1)/2} = \sum_{m \geqq 0} \frac{q^{m^2}}{(q;q)_m}. \tag{2.11}$$

By Jacobi's triple product (2.1), we directly deduce from (2.11) that

$$\frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty} = \sum_{m \geqq 0} \frac{q^{m^2}}{(q;q)_m}, \tag{2.12}$$

which is the well-known first Rogers–Ramanujan identity [6; p. 36].

The above proof differs slightly from Rogers' version. He replaced z by $e^{i\theta}$ and wrote everything in terms of cosines. In this way the sum in (2.8) would have had $0 \leqq j \leqq m$ instead of $-m \leqq j \leqq m$.

We shall exploit the bilateral nature of (2.8) in our developments in Section 3. However, it is important to observe that several authors, notably Paule [9] followed by Berkovich, McCoy, and Schilling [4], have recognized that bilateral sums are quite valuable in extensions of the work of Rogers.

The most important idea to be grasped here is simply this. Rogers' second proof of the Rogers–Ramanujan identities consists purely of an umbral map applied to a classical identity from elliptic theta functions, in this case Jacobi's triple product identity.

There are plenty of other identities of this nature, e.g., the quintuple product [6; p. 134], Winquist's identity [13], the Macdonald identities [8], [12], etc. In the interests of brevity we follow one obvious path in the next section, namely the product of independent instances of Jacobi's triple product. This will set us up for the Pentagonal Number Theorems proved in Section 5.

3. MULTI-DIMENSIONAL BAILEY CHAINS

In this section we shall follow up on just one possibility suggested by the discussion at the end of Section 2. Namely, let us look at the same idea applied to the product of s independent copies of Jacobi's triple product:

$$\prod_{j=1}^{s} \sum_{n_j=-\infty}^{\infty} z_j^{n_j} q^{n^2_j} = \prod_{j=1}^{s} (q^2; q^2)_\infty (-z_j q; q^2)_\infty (-z_j^{-1} q; q^2)_\infty]. \tag{3.1}$$

Precisely s copies of the umbral treatment of the case $s=1$ can be carried out, and one will obtain an s-fold version of (2.7) and (2.8). Having made this initial observation, we can jump directly to a full s-fold extension of Bailey's lemma:

THEOREM 1. *If for* $n_1, n_2, ..., n_s \geqq 0$,

$$\beta_{n_1, n_2, ..., n_s} = \sum_{r_1=-\infty}^{n_1} \sum_{r_2=-\infty}^{n_2} \cdots \sum_{r_s=-\infty}^{n_s} \frac{\alpha_{r_1, r_2, ..., r_s}}{\prod_{j=1}^{s} (q; q)_{n_j - r_j} (a_j q; q)_{n_j + r_j}} \tag{3.2}$$

then

$$\beta'_{n_1, n_2, ..., n_s} = \sum_{r_1=-\infty}^{n_1} \sum_{r_2=-\infty}^{n_2} \cdots \sum_{r_s=-\infty}^{n_s} \frac{\alpha'_{r_1, r_2, ..., r_s}}{\prod_{j=1}^{s} (q; q)_{n_j - r_j} (a_j q; q)_{n_j + r_j}}, \tag{3.3}$$

where

$$\alpha'_{r_1, r_2, ..., r_s} = \left(\prod_{j=1}^{s} \frac{(\rho_j)_{r_j} (\sigma_j)_{r_j} (q_j q/(\rho_j \sigma_j))^{r_j}}{(a_j q/\rho_j)_{r_j} (a_j q/\sigma_j)_{r_j}} \right) \alpha_{r_1, r_2, ..., r_s} \tag{3.4}$$

and

$$\beta'_{n_1, n_2, ..., n_s} = \prod_{j=1}^{s} \sum_{m_j=-\infty}^{n_j} \frac{(\rho_j)_{m_j} (\sigma_j)_{m_j} (a_j; q/(\rho_j \sigma_j))_{n_j - m_j} (\frac{a_j; q}{\rho_j \sigma_j})^{m_j} \beta_{m_1, ..., m_s}}{(q)_{n_j - m_j} (a_j q/\rho_j)_{n_j} (a_j q/\sigma_j)_{n_j}}. \tag{3.5}$$

When $s=1$, this is the bilateral Bailey pair formula given in [4; p. 48]. The proof of this formula is precisely the proof of the $s=1$ case now repeated independently s times. Given this fact, we omit the details.

In the next section, we shall only be interested in the limiting case of Theorem 1, in which $a_1 = a_2 = \cdots = a_s = 1$, and each of $n_1, n_2, ..., n_s$, $\rho_1, \rho_2, ..., \rho_s, \sigma_1, \sigma_2, ..., \sigma_s \to +\infty$. This provides what might be termed the Weak Multi-dimensional Bailey Lemma. It is the natural multi-dimensional analog of (2.7) and (2.8).

COROLLARY 1. *If for* $n_1, n_2, ..., n_s \geqq 0$

$$\beta_{n_1, n_2, ..., n_s} = \sum_{r_1=-n_1}^{n_1} \sum_{r_2=-n_2}^{n_2} \cdots \sum_{r_s=-n_s}^{n_s} \frac{\alpha_{r_1, r_2, ..., r_s}}{\prod_{j=1}^{s} (q; q)_{n_j - r_j} (q; q)_{n_j + r_j}}, \tag{3.6}$$

then

$$\sum_{n_1, \ldots, n_s \geqq 0} q^{n^2{}_1 + n^2{}_2 + \cdots + n^2{}_s} \beta_{n_1, n_2, \ldots, n_s}$$

$$= \frac{1}{(q;q)_\infty^s} \sum_{m_1, m_2, \ldots, m_s = -\infty}^{\infty} q^{m^2{}_1 + m^2{}_2 + \cdots + m_s^2} \alpha_{m_1, m_2, \ldots, m_s}. \tag{3.7}$$

4. PENTAGONAL NUMBER THEOREMS

Our object here is to provide some striking applications of our work in Section 2. We shall prove three results of increasing difficulty.

THEOREM 2. *Identity* (1.2) *is valid.*

Proof. We apply Corollary 1 with $s = 2$ and

$$\alpha_{m_1, m_2} = (-1)^{m_1 + m_2} q^{\binom{m_1 + m_2}{2}}. \tag{4.1}$$

Consequently

$$(q;q)_{2n_1} (q;q)_{2n_2} \beta_{n_1, n_2}$$

$$= \sum_{r_1 = -n_1}^{n_1} \sum_{r_2 = -n_2}^{n_2} \begin{bmatrix} 2n_1 \\ n_1 - r_1 \end{bmatrix} \begin{bmatrix} 2n_2 \\ n_2 - r_2 \end{bmatrix} (-1)^{r_1 + r_2} q^{\binom{r_1 + r_2}{2}}, \tag{4.2}$$

where

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{cases} 0, & \text{if } \quad B < 0 \text{ or } B > A \\ \frac{(q;q)_A}{(q;q)_B (q;q)_{A-B}} & \text{otherwise.} \end{cases} \tag{4.3}$$

By symmetry we may assume $n_2 \geqq n_1$, and by the q-binomial theorem [1; p. 36, Eq. (3.3.6)], we deduce that

$$(q;q)_{2n_1} (q;q)_{2n_2} \beta_{n_1 n_2}$$

$$= (-1)^{n_2} \sum_{r_1 = -n_1}^{n_1} \begin{bmatrix} 2n_1 \\ n_1 - r_1 \end{bmatrix} (-1)^{r_1} q^{\binom{r_1}{2} - r_1 n_2 + \binom{n_2}{2}} (q^{r_1 - n_2}; q)_{2n_2}$$

$$= \begin{cases} 0 & \text{if } \quad n_2 \neq n_1 \\ (q;q)_{2n_1} & \text{if } \quad n_1 = n_2. \end{cases} \tag{4.4}$$

Consequently

$$\beta_{n_1, n_2} = \begin{cases} 0 & \text{if} \quad n_1 \neq n_2 \\ \frac{1}{(q;q)_{2n_1}} & \text{if} \quad n_1 = n_2. \end{cases} \tag{4.5}$$

Theorem 2 now follows immediately from (4.1), (4.5) and Corollary 1. ∎

Our next result is harder to prove than Theorem 2 primarily because we introduce a small asymmetry in the definition of α_{m_1, m_2}.

THEOREM 3.

$$\sum_{m, n \geqq 0} \frac{(-1)^n q^{m^2+2mn+2n^2}(-1;q)_m}{(q;q^2)_{m+n}(q^2;q^2)_n(q;q)_m}$$
$$= \frac{\sum_{i,j=-\infty}^{\infty} (-1)^j q^{i(3i-1)/2 + j(3j-1)/2 + ij}}{(q;q)_\infty^2}. \tag{4.6}$$

Proof. We apply Corollary 1 with $s = 2$ and

$$\alpha_{m_1, m_2} = (-1)^{m_2} q^{\binom{m_1+m_2}{2}}. \tag{4.7}$$

Consequently

$$(q;q)_{2n_1}(q;q)_{2n_2}\beta_{n_1,n_2}$$
$$= \sum_{r_1=-n_1}^{n_1} \sum_{r_2=-n_2}^{n_2} \begin{bmatrix} 2n_1 \\ n_1+r_1 \end{bmatrix} \begin{bmatrix} 2n_2 \\ n_2+r_2 \end{bmatrix} (-1)^{r_2} q^{\binom{r_1+2}{2}}. \tag{4.8}$$

By the q-binomial theorem we find

$$(q;q)_{2n_1}(q;q)_{2n_2}\beta_{n_1,n_2}$$
$$= \sum_{r_1=-n_1}^{n_1} \begin{bmatrix} 2n_1 \\ n_1+r_1 \end{bmatrix} q^{\binom{r_1}{2}}(q^{1-r_1};q)_{n_2}(q^{r_1};q)_{n_2}. \tag{4.9}$$

Now the only non-vanishing terms in (4.9) occur for $n_2 < r_1 \leqq n_1$ and $-n_1 \leqq r_1 \leqq -n_2$. Therefore

$$(q;q)_{2n_1}(q;q)_{2n_2}\beta_{n_1,n_2}$$
$$= \sum_{r_1 > n_2} \begin{bmatrix} 2n_1 \\ n_1+r_1 \end{bmatrix} q^{\binom{r_1}{2}}(q^{1-r_1};q)_{n_2}(q^{r_1};q)_{n_2}$$
$$+ \sum_{r_1 \geqq n_2} \begin{bmatrix} 2n_1 \\ n_1-r_1 \end{bmatrix} q^{\binom{-r_1}{2}}(q^{r_1+1};q)_{n_2}(q^{-r_1};q)_{n_2}$$

$$= -q^{n_2} \sum_{r_1 \geqq 0} \begin{bmatrix} 2n_1 \\ n_1 + r_1 \end{bmatrix} q^{\binom{r_1}{2} - r_1} (q^{1-r_1}; q)_{n_2 - 1} (q^{r_1}; q)_{n_2} (1 - q^{2r_1})$$

(by combining the two sums)

$$= -q^{n_2 - 1} \sum_{r_1 \geqq 0} \begin{bmatrix} 2n_1 \\ n_1 + r_1 + 1 \end{bmatrix} q^{\binom{r_1}{2}} (q^{-r_1}; q)_{n_2 - 1} (q^{r_1 + 1}; q)_{n_2} (1 - q^{2r_1 + 2})$$

(shifting r_1 to $r_1 + 1$)

$$= (-1)^{n_2} (q; q)_{2n_2} \begin{bmatrix} 2n_1 \\ n_1 + n_2 \end{bmatrix}$$

$$\times {}_6\phi_5 \left(\begin{matrix} q^{2n_2}, q^{n_2+1}, -q^{n_2+1}, q^{n_2-n_1}, q^{n_2+1/2}, -q^{n_2+1/2}; q, -q^{n_1-n_2} \\ q^{n_2}, -q^{n_2}, q^{n_2+n_1+1}, q^{n_2+1/2}, -q^{n_2+1/2} \end{matrix} \right)$$

(in the notation of [6; p. 4])

$$= (-1)^{n_2} \begin{bmatrix} 2n_1 \\ n_1 + n_2 \end{bmatrix} \frac{(q)_{n_1+n_2} (-1; q)_{n_1-n_2}}{(q^{2n_2+1}; q^2)_{n_1-n_2}}$$

(by [6; p. 238, Eq. (II.21)])

$$= (-1)^{n_2} (-q; q)_{n_1} (-1; q)_{n_1-n_2} (q; q^2)_{n_2} (q^{n_1-n_2+1}; q)_{n_2}. \tag{4.10}$$

Note that if $n_2 > n_1$, then β_{n_1, n_2} must be 0. Consequently by Corollary 1,

$$\sum_{n_1 \geqq n_2} \beta_{n_1, n_2} q^{n_1^2 + n_2^2} = \frac{\sum_{i, j=-\infty}^{\infty} (-1)^j q^{i^2 + j^2 + \binom{i+j}{2}}}{(q; q)_\infty^2}. \tag{4.11}$$

Shifting the index n_1 to $n_1 + n_2$ and replacing β_{n_1, n_2} by the expression found for it in (4.10) we obtain (4.6), thus proving Theorem 3. ∎

Our final theorem, the proof of (1.3), requires an initial inversion lemma.

LEMMA 1. *If two sequences* $\{x_n\}_{n=0}^{\infty}$ *and* $\{\rho_n\}_{n=0}^{\infty}$ *satisfy either of the following two relations for all n, then they satisfy both.*

$$x_n = \sum_{j=0}^{n} (-1)^j \begin{bmatrix} n+j \\ n-j \end{bmatrix} q^{\binom{j+1}{2} - nj} \rho_j, \tag{4.12}$$

and

$$\rho_n = \sum_{k=0}^{n} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} 2n+1 \\ n+k+1 \end{bmatrix} \frac{(1-q^{2k+1})}{(1-q^{2n+1})} x_k. \tag{4.13}$$

Proof. This is merely a special case of the inversion theorem for Bailey pairs [2; p. 278] with $a=q$, $\beta_n=\rho_n/(q^2;q)_{2n-1}$, and $\alpha_n=(-1)^n q^{\binom{n}{2}}(1-q^{2n+1})\, x_n$. ∎

THEOREM 4. *Identity* (1.3) *is valid.*

Proof. We apply Corollary 1 with $s=3$ and

$$\alpha_{m_1, m_2, m_3}=(-1)^{m_1+m_2+m_3}\, q^{\binom{m_1+m_2+m_3}{2}}. \tag{4.14}$$

If we can prove that

$$\sum_{i=-n_1}^{n_1}\sum_{j=-n_2}^{n_2}\sum_{k=-n_3}^{n_3}(-1)^{i+j+k}\, q^{\binom{i+j+k}{2}}\begin{bmatrix}2n_1\\ n_1+i\end{bmatrix}\begin{bmatrix}2n_2\\ n_2+j\end{bmatrix}\begin{bmatrix}2n_3\\ n_3+k\end{bmatrix}$$
$$=\frac{(q;q)_{2n_1}(q;q)_{2n_2}(q;q)_{2n_3}}{(q)_{n_1+n_2-n_3}(q;q)_{n_1+n_3-n_2}(q;q)_{n_2+n_3-n_1}}, \tag{4.15}$$

then Corollary 1 implies directly that identity (1.3) holds. So to conclude our proof we must establish (4.15).

To this end, we define

$$S(n_1, n_2, k)$$
$$=\frac{1}{(q;q)_{2n_1}}\sum_{i=-n_1}^{n_1}\sum_{j=-n_2}^{n_2}(-1)^{i+j}\, q^{\binom{i+j}{2}+k(i+j)}\begin{bmatrix}2n_1\\ n_1+i\end{bmatrix}\begin{bmatrix}2n_2\\ n_2+j\end{bmatrix}, \tag{4.16}$$

and

$$T(n_1, n_2, n_3)=\frac{(q;q)_{2n_2}}{(q;q)_{n_1+n_2-n_3}}\begin{bmatrix}2n_3\\ n_1+n_3-n_2\end{bmatrix}. \tag{4.17}$$

Then we may rewrite (4.15) as

$$\sum_{k=-n_3}^{n_3}(-1)^k q^{\binom{k}{2}} S(n_1, n_2, k)\begin{bmatrix}2n_3\\ n_3+k\end{bmatrix}=T(n_1, n_2, n_3). \tag{4.18}$$

Next we note that by replacing i by $-i$ and j by $-j$ in (4.16), we have

$$S(n_1, n_2, k)=S(n_1, n_2, 1-k). \tag{4.19}$$

Using (4.19) to combine positive and negative terms in (4.18), we may rewrite (4.18) as

$$q^{n_3} \sum_{k=0}^{n_3} (-1)^k q^{\binom{k}{2}} S(n_1, n_2, k+1) \begin{bmatrix} 2n_3+1 \\ n_3+k+1 \end{bmatrix} \frac{(1-q^{2k+1})}{(1-q^{2n_3+1})}$$

$$= T(n_1, n_2, n_3). \tag{4.20}$$

We now apply our lemma with $n = n_3$, $\rho_n = q^{-n} T(n_1, n_2, n)$ and $x_k = S(n_1, n_2, k+1)$. So we see that (4.15) is actually equivalent to

$$S(n_1, n_2, n+1) = \sum_{j=0}^{n} (-1)^j \begin{bmatrix} n+j \\ n-j \end{bmatrix} q^{\binom{j+1}{2} - nj} q^{-j} T(n_1, n_2, j). \tag{4.21}$$

We have now reduced the proof of Theorem 4 to proving that (4.21) is true. Furthermore, owing to the symmetry in (1.3), we may without loss of generality assume that $n_1 \geqq n_2 \geqq n_3$. The right-hand side of (4.21) when reduced to q-hypergeometric notation [6; p. 4] is, in fact, equal to

$$\frac{(-1)^n (q;q)_{2n} (q;q)_{2n_2} q^{-\binom{n+1}{2}}}{(q;q)_{n_1+n_2-n} (q;q)_{n_1+n-n_2} (q;q)_{n_2+n-n_1}} \, {}_3\phi_2 \left(\begin{matrix} q^{-n_1+n_2-n}, q^{-n_2+n_1-n}, 0; q; q \\ q^{n_1+n_2-n+1}, q^{-2n} \end{matrix} \right). \tag{4.22}$$

Now we can simplify the expression for $S(n_1, n_2, k)$ given in (4.16) by applying the q-binomial theorem [6; p. 236, Eq. (II.4)]

$$\begin{aligned} S(n_1, n_2, k) &= \frac{1}{(q;q)_{2n_1}} \sum_{j=-n_1-n_2}^{n_2-n_1} (-1)^j q^{\binom{j}{2}+kj} \begin{bmatrix} 2n_2 \\ n_2+n_1+j \end{bmatrix} (q^{k+j};q)_{2n_1} \\ &= (-1)^{n_1+n_2} q^{\binom{n_1+n_2+1}{2} - k(n_1+n_2)} \\ &\quad \times \frac{1}{(q;q)_{2n_1}} \sum_{j \geqq 0} \frac{(q^{-2n_2}, q)_j}{(q;q)_j} \frac{(q^{k-n_2-n_1};q)_{2n_1+j}}{(q^{k-n_2-n_1};q)_j} q^{j(k+n_2-n_1)} \\ &= \frac{(-1)^{n_1+n_2} q^{\binom{k-n_2-n_1}{2} - \binom{k}{2}} (q^{k-n_2-n_1};q)_{2n_1}}{(q;q)_{2n_1}} \\ &\quad \times {}_2\phi_1 \left(\begin{matrix} q^{-2n_2}, q^{k-n_2+n_1}; q, q^{k+n_2-n_1} \\ q^{k-n_2-n_1} \end{matrix} \right) \quad \text{(in the notation of [6; p. 4]).} \end{aligned} \tag{4.23}$$

Therefore

$$
\begin{aligned}
S(n_1, n_2, n+1) &= S(n_1, n_2, -n) \qquad \text{(by (4.19))} \\
&= \frac{(-1)^{n_1+n_2} q^{\binom{n_1+n_2+n+1}{2}-\binom{n+1}{2}} (q^{-n_1-n_2-n}; q)_{2n_1}}{(q; q)_{2n_1}} \\
&\quad {}_2\phi_1\left(\begin{matrix} q^{-2n_2}, q^{-n-n_2+n_1}; q, q^{n_2-n-n_1} \\ q^{-n_1-n_2-n} \end{matrix}\right) \\
&= (-1)^{n_1+n_2} q^{\binom{n_1-n_2}{2}+n^2_2+nn_2-n^2_1+nn_1-n^2} \begin{bmatrix} 2n \\ n_2+n-n_1 \end{bmatrix} \\
&\quad {}_2\phi_1\left(\begin{matrix} q^{-n-n_2+n_1}, q^{-n-n_2+n_1}; q; q^{-2n_1} \\ q^{-2n} \end{matrix}\right) \quad \text{(by [6; p. 240, Eq. (III.2)])}
\end{aligned} \tag{4.24}
$$

Finally then, the proof of Theorem 4 is reduced to proving that the expression in (4.22) is equal to the right-hand side of (4.24). So what we must prove is

$$
\begin{aligned}
&{}_3\phi_2\left(\begin{matrix} q^{-n_1+n_2-n}, q^{-n_2+n_1-n}, 0; q, q \\ q^{n_1+n_2-n+1}, q^{-2n} \end{matrix}\right) \\
&\quad = \frac{(-1)^{n+n_1+n_2} q^{\binom{n_1-n_2}{2}+n_2^2+nn_2-n_1^2+nn_1-n^2+\binom{n+1}{2}} (q; q)_{n_1+n_2-n}}{(q; q)_{2n_2}} \\
&\qquad \times {}_2\phi_1\left(\begin{matrix} q^{-n-n_2+n_1}, q^{-n-n_2+n_1}; q, q^{-2n_1} \\ q^{-2n} \end{matrix}\right).
\end{aligned} \tag{4.25}
$$

But this is just [6; Eq. (III.11), p. 241] with $c=0$, $d=q^{-2n}$, $e=q^{n_1+n_2-n+1}$, $b=q^{-n_1+n_2-n}$, and the n of (III.11) replaced by $n+n_2-n_1$. Therefore since (4.25) is true we see that in fact Theorem 4 is proved. ▮

5. CONCLUSION

There are several important summary observations. First of all, it would be very nice to have a simpler proof of the polynomial identity (4.15). It is rather surprising that (4.15) is so difficult in light of the fact that the just slightly simpler (4.4) is very easy. In fact, using a q-version of K. Wegschaider's "MultiSum" package, P. Paula and A. Riese derived a recurrence proof of (4.15); see also http://www.risc.uni-linz.ac.at/research/combinat/risc/.

It is natural to ask for a nice q-series expansion for an s-dimensional Pentagonal Number series, namely

$$\frac{\sum_{n_1, n_2, \ldots, n_s} (-1)^{n_1+n_2+\cdots+n_2} q^{n_1^2+n_2^2+\cdots+n_s^2+\binom{n_1+n_2+\cdots+n_s}{2}}}{(q;q)_\infty^s}. \tag{5.1}$$

Finally and most important, the object here was to observe that the Bailey chain could be viewed as arising from the application of an umbral operator to a classical elliptic function identity, namely Jacobi's Triple Product Identity. There are many such identities, such as the Quintuple Product Identity [6; p. 134], Winquist's Identity [13], or the Macdonald identities [8, 12] in general. Whether comparable theorem applications exist related to these identities remains to be seen.

REFERENCES

1. G. E. Andrews, "The Theory of Partitions," Encyclopedia of Mathematics and Its Applications, Vol. 2 (G.-C. Rota, Ed.), Addison–Wesley, Reading, MA, 1976. Reprinted: Cambridge Univ. Press, Cambridge, UK, 1998.
2. G. E. Andrews, Multiple series Rogers–Ramanujan type identities, *Pac. J. Math.* **141** (1984), 267–283.
3. G. E. Andrews, "q-Series, Their Development and Application in Analysis, Number Theory, Combinatorics, Physics and Computer Algebra," CBMS Regional Conference Lecture Series, Vol. 66, Amer. Math. Soc., Providence, RI, 1986.
4. A. Berkovich, B. M. McCoy, and A. Schilling, $N=2$ supersymmetry and Bailey pairs, *Phys. A* **228** (1996), 33–62.
5. L. E. Dickson, "History of the Theory of Numbers," Vol. 2, Carnegie Inst. of Washington, Publ. 256, 1920. Reissued: Chelsea, New York, 1952.
6. G. Gasper and M. Rahman, "Basic Hypergeometric Series," Encyclopedia of Mathematics and Its Applications, Vol. 35 (G.-C. Rota, Ed.), Cambridge Univ. Press, Cambridge, UK, 1990.
7. G. Humbert, Démonstration analytique d'une formule de Liouville, *Bull. Sci. math.* (*2*) **34** (1910), 29–31.
8. I. G. Macdonald, Affine root systems and Dedekind's η-function, *Invent. Math.* **15** (1972), 91–143.
9. P. Paule, The concept of Bailey chains, *Sém. Lothar. Combin. B* **18f** (1987), 24.
10. L. J. Rogers, On two theorems of combinatory analysis and some allied identities, *Proc. London Math. Soc.* (*2*) **16** (1917), 315–336.
11. G. C. Rota, Ten mathematics problems I will never solve, *DMV Mitteilungen* **2** (1998), 45–52.
12. D. Stanton, An elementary approach to the Macdonald identities, *in* "q-Series and Partitions" (D. Stanton, Ed.), IMA Volumes in Mathematics and Its Applications, Vol. 18, Springer-Verlag, Berlin, 1989.
13. L. Winquist, An elementary proof of $p(11m+6) \equiv 0 \pmod{11}$, *J. Combin. Theory* **6** (1969), 56–59.